L.2 4.09.2008
SB

① Vector field

$$M \ni p \xmapsto{\;X\;} X_p \in T_p(M)$$

i.e. a vector field is an assignment to every point $p \in M$ of a vector X_p from $T_p(M)$.

$\mathcal{F}(M)$ – algebra of C^∞ functions on M

one can think about X as a map

$$X: \mathcal{F}(M) \longrightarrow \mathcal{F}(M)$$
$$f \longmapsto X(f)$$

$$X(f)(p) = X_p(f)$$

X is <u>of class C^k</u> if $\forall f \in \mathcal{F}(M)$ $X(f)$ is of class C^k

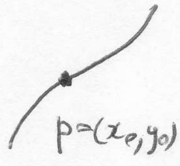
locally (U, x): $X = X^\mu \dfrac{\partial}{\partial x^\mu}$ where $X^\mu : M \longrightarrow \mathbb{R}^n$

if X is of class C^k then $X^\mu = X^\mu(p)$ are of class C^k and vice versa.

$\mathfrak{X}(M)$ – vector space of vector fields of class C^∞ on M (infinite dimensional!)

② <u>Trajectory of a vector field</u> passing through p

or <u>integral curve of X</u> passing through p



<u>Ex</u> $X = y\partial_x - x\partial_y$ vector field on $\mathbb{R}^2$

$X^\mu = \begin{pmatrix} y \\ -x \end{pmatrix}$ $\gamma^\mu(t) = \begin{pmatrix} x \\ y \end{pmatrix}(t)$ s.t. $\dfrac{d\gamma^\mu}{dt} = X^\mu$

$$\left.\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \frac{d(x+iy)}{dt} &= -i(x+iy) \\ x+iy &= (x_0+iy_0)e^{-it} \end{aligned}\right\} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix}(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\gamma^\mu(t) = \varphi_t \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$\varphi_t = \begin{pmatrix} \cos t & +\sin t \\ -\sin t & \cos t \end{pmatrix}$ $\varphi_0 = id$

$\varphi_{t+t'} = \varphi_t \circ \varphi_{t'}$

Ex $\quad X = \partial_y - x^{-2}\partial_y \qquad$ on $\quad \mathbb{R}_+ \setminus \{0\}$

$$\frac{d\gamma^\mu}{dt} = X^\mu \qquad \left.\begin{array}{l} \dfrac{dx}{dt} = 1 \\ \dfrac{dy}{dt} = -x^{-2} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x = t + x_0 \\ y = \dfrac{1}{t+x_0} + y_0 - \dfrac{1}{x_0} \end{array}\right.$$

$$\gamma^\mu(t) = \varphi_t\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} t + x_0 \\ \dfrac{1}{t+x_0} + y_0 - \dfrac{1}{x_0} \end{pmatrix}$$

$$\varphi_0\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \qquad \underbrace{\varphi_{t+t'} = \varphi_t \circ \varphi_{t'}}$$

this only holds locally.

1-parameter transformation groups of M	Local 1 par. tr. groups of M
$\mathbb{R} \times M \ni (t,p) \xrightarrow{\text{smooth}} \varphi_t(p) \in M$	$\forall_p \ \exists \ U_p, \varepsilon > 0$
1° $\varphi_0 = id$	$]-\varepsilon,\varepsilon[\times U_p \ni (t,p) \to \varphi_t(p) \in M$
2° $\varphi_t \circ \varphi_{t'} = \varphi_{t+t'}$	1°
	2° as above for $\lvert t\rvert, \lvert t'\rvert, \lvert t+t'\rvert < \varepsilon$

If t and t' are to far from 0 the second component blows up.

Def

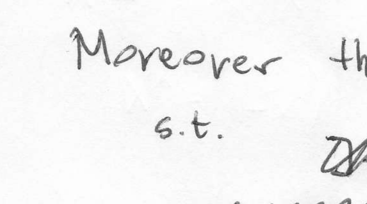
$t \longrightarrow \gamma(t)$ is called an integral curve of X passing through p if $\gamma(0) = p$,

↑ differentiable

$$\frac{d\gamma}{dt} = X_{\gamma(t)}$$

$\rightsquigarrow$ locally $\quad \dfrac{dx^\mu}{dt} = X^\mu(x^\nu(t)) \quad$ where $\quad X = X^\mu(x^\alpha)\dfrac{\partial}{\partial x^\mu}$

~~a uniqu~~ theory of autonomous systems of ODEs.



There exist ~~[illegible]~~ $\varepsilon > 0$ ~~[illegible]~~ and a unique curve

$$]-\varepsilon,\varepsilon[\ \ni t \longmapsto \gamma(t) = \varphi(t,p) = \varphi_t(p)$$

s.t. $\quad \gamma(0) = p$ and $\dfrac{d\gamma}{dt} = X_{\gamma(t)}$.

Moreover there exists a neighbourhood $U_p \subset M$ s.t.

$\cancel{]-\varepsilon,\varepsilon[\times U_p \ni (t,p) \longrightarrow \varphi_t(p)}$

~~[illegible]~~

if $\quad t \in]-\varepsilon,\varepsilon[\quad$ then $\quad \varphi_t : p \longrightarrow \varphi_t(p)$ is diffeo

~~[illegible]~~ with the properties

1) $\varphi_0 = id$

2) $\varphi_{t+t'} = \varphi_t \circ \varphi_{t'}$

when $t, t' \in]-\varepsilon,\varepsilon[$.

$\boxed{\varphi_t \text{ is called a } \underline{\text{flow}} \text{ of } X}$

Of course $X \in \mathfrak{X}(M)$ then the map

$$X : \mathcal{F}(M) \longrightarrow \mathcal{F}(M) \text{ is}$$

1) linear
2) satisfies $X(f \cdot g) = X(f) \cdot g + f \cdot X(g)$

③ <u>Commutator</u>

$$\mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X, Y) \longmapsto [X, Y] \in \mathfrak{X}(M)$$

$$[X, Y](f) = X(Y(f)) - Y(X(f))$$

obviously $[X, Y] : \mathcal{F}(M) \longrightarrow \mathcal{F}(M)$ and is linear

but also $[X, Y](f \cdot g) = [X, Y](f) \cdot g + f [X, Y](g)$ check!

- locally $\quad X = X^\mu \partial_\mu \quad , \; Y = Y^\mu \partial_\nu$

$$[X, Y] = \left(X^\mu Y^\nu{}_{,\mu} - Y^\mu X^\nu{}_{,\mu} \right) \partial_\nu$$

- <u>properties</u>

1° $[\,,\,]$ - bilinear

2° $[X, Y] = -[Y, X]$ antisymmetric

3° $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$, Jacobi.

$\left(\mathfrak{X}(M), [\,,\,] \right)$ Lie algebra of smooth vector fields over M (infinite dim).

4 (Vector) distribution

Def An m-dimensional distribution S on M is a map

$$M \ni p \xrightarrow{S} S_p \subset T_p(M)$$

S_p: m-dimensional (vector) subspace of $T_p(M)$

if $\forall p \in M \; \exists \, U_p \; \exists \, (X_i)_{i=1,\dots m}$ of class C^∞ s.t.

$\forall q \in U_p \; \left(X_i\big|_q \right)$ is a basis for $S_q \Rightarrow S_q$ <u>is smooth</u>

<u>Only SMOOTH distributions from now on.</u>

- $X \in S \iff \forall_p \; X_p \in S_p$

- S is <u>involutive</u> $\iff X, Y \in S \Rightarrow [X, Y] \in S$

- M_S is an integral manifold of S iff
  - M_S is a submanifold of $\underline{M}$ s.t.

$$\forall p \in M_S \quad T_p(M_S) = S_p$$

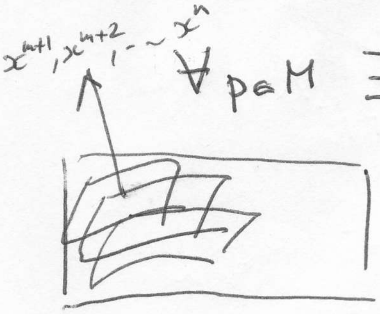
Note: Vector field is a distribution of dim 1.
Its integral manifolds $\equiv$ integral curves.

What about existence of integral manifolds for $m > 1$ dimensional distributions?

⑤ Frobenius theorem

$\left(S \text{ is involutive} \right) \iff$ (through every point $p \in M$ passes precisely one (maximal) integral manifold M_S of S)

in such a case



$\forall p \in M \;\exists (\mathcal{U}, x)$ s.t. $p \in \mathcal{U}$ and ~~intersections of~~ all $E_S \cap \mathcal{U}$ are given by $x^{m+1}, x^{m+2}, \ldots x^n = \text{const}$.

then

$$X_i = A^j{}_i(x^\mu) \frac{\partial}{\partial x^j} \qquad j = 1, \ldots m$$

is a local basis in S

Fact

X_i on $\mathcal{U}$ linearly independent vector fields.

$[X_i, X_j] = 0 \quad \forall i,j = 1, \ldots m \iff$ there exists a coordinate system x^μ in $V \subset \mathcal{U}$ s.t. $X_i = \frac{\partial}{\partial x^i}$, $i = 1, \ldots m$.

⑥ Tensors.

$n < \infty$

V – n-dimensional vector space over $\mathbb{K} = \mathbb{R}, \mathbb{C}$

$$V^* = \{ \omega : V \xrightarrow[\text{linear}]{} \mathbb{K} \}$$

$$V^r_s = \underbrace{V \otimes \dots \otimes V}_{r} \otimes \underbrace{V^* \otimes \dots \otimes V^*}_{s} =$$

$$= L(\underbrace{V^*, V^*, \dots, V^*}_{r}, \underbrace{V, \dots V}_{s}; \mathbb{R})$$

$\uparrow$

multilinear maps fra $V^* \times V^* \times \dots V^* \times V \times \dots V \longrightarrow \mathbb{R}$.

$\{e_\mu\}$ – basis in V
$\mu = 1, \dots n$

$\{e^\mu\}$ – dual basis in V^* defined by
$\mu = 1, \dots n$

$$e^\mu(e_\nu) = \delta^\mu{}_\nu$$

$$V \ni v = v^\mu e_\mu \, ; \quad \omega = \omega_\mu e^\mu \in V^*$$

Basis in V^r_s:

$$e_{\mu_1} \otimes \dots e_{\mu_r} \otimes e^{\nu_1} \otimes \dots e^{\nu_s} \text{ defined by}$$

$$\left(e_{\mu_1} \otimes \dots \otimes e_{\mu_r} \otimes e^{\nu_1} \otimes \dots \otimes e^{\nu_s}\right)(e^{\alpha_1}, \dots e^{\alpha_r}, e_{\beta_1}, \dots e_{\beta_s}) =$$

$$= \delta_{\mu_1}^{\alpha_1} \dots \delta_{\mu_r}^{\alpha_r} \delta_{\beta_1}^{\nu_1} \dots \delta_{\beta_s}^{\nu_s}$$

$$V^r_s \ni K = K^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} e_{\mu_1} \otimes \dots \otimes e_{\mu_r} \otimes e^{\nu_1} \otimes \dots \otimes e^{\nu_s}$$

Contraction $\quad C^i_j : V^r_s \xrightarrow{\text{linear}} V^{r-1}_{s-1}$

$1 \le i \le r$
$1 \le j \le s$

$$C^i_j(v_1 \otimes \dots \otimes v_r \otimes \omega^1 \otimes \dots \otimes \omega^s) = $$

$$= \langle \omega^j, v_i \rangle \, v_1 \otimes \dots \overset{i}{\check{}} \dots \otimes v_r \otimes \omega^1 \otimes \dots \overset{j}{\check{}} \dots \otimes \omega^s$$

(7) Change of the basis

$$\boxed{e'^{\mu} = a^{\mu}{}_{\nu} e^{\nu}} \qquad e' = a e$$

$$a = (a^{\mu}{}_{\nu}) \in GL(n, \mathbb{K})$$

$$e'_{\mu} = e_{\nu} b^{\nu}{}_{\mu}$$

$$\underset{\delta^{\mu}{}_{\nu}}{\underset{\shortparallel}{e'^{\mu}(e'_{\nu})}} = a^{\mu}{}_{\rho} b^{\sigma}{}_{\nu} e^{\rho}(e_{\sigma}) = a^{\mu}{}_{\rho} b^{\rho}{}_{\nu}$$

$$\Rightarrow \quad a \cdot b = 1 \Rightarrow b = a^{-1}$$

$$\boxed{e'_{\mu} = e_{\nu} a^{-1\nu}{}_{\mu}}$$

$$v = v^{\mu} e_{\mu} = v'^{\mu} e'_{\mu} = v'^{\mu} e_{\nu} a^{-1\nu}{}_{\mu}$$

$$\Rightarrow v^{\nu} = v'^{\mu} a^{-1\nu}{}_{\mu} \Rightarrow \boxed{v'^{\mu} = a^{\mu}{}_{\nu} v^{\nu}}$$

$$\omega = \omega_{\mu} e^{\mu} = \omega'_{\mu} a^{\mu}{}_{\nu} e^{\nu}$$

$$\Rightarrow \omega_{\nu} = \omega'_{\mu} a^{\mu}{}_{\nu} \Rightarrow \boxed{\omega'_{\mu} = \omega_{\nu} a^{-1\nu}{}_{\mu}}$$

$$K^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} \longmapsto a^{\mu_1}{}_{\alpha_1} \dots a^{\mu_r}{}_{\alpha_r} K^{\alpha_1 \dots \alpha_r}{}_{\beta_1 \dots \beta_s} a^{-1\beta_1}{}_{\nu_1} \dots a^{-1\beta_s}{}_{\nu_s}$$

$$= K'^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}.$$

~~old style definition of tensor:~~

~~$T = P(V) \times W$~~

~~vector space~~

Old style view on tensors

$$K \sim (e, K^{\alpha_1 \dots \alpha_r}{}_{\beta_1 \dots \beta_s})$$

$$e \mapsto e' = e a^{-1}$$

$$K^{\alpha_1 \dots \alpha_r}{}_{\beta_1 \dots \beta_s} \longmapsto K'^{\mu_1 \dots \mu_s}{}_{\nu_1 \dots \nu_s}$$

How to introduce a vector space structure on such pairs?

(8) Objects of type ρ.

$$\rho : G \xrightarrow{hom} GL(W) \quad \text{i.e. } \rho \text{ is a representation of } G \text{ in } W.$$

vector space of $\dim W = N < \infty$

$a \in G$; $\quad W \ni w \xrightarrow{\rho(a)} \rho(a)w \in W$ is a left action of G on W.

- $\rho(1) = id$
- $\rho(a \cdot b) = \rho(a)\rho(b)$

V another vector space of $\dim V = n < \infty$; $P(V)$ set of all basis in V

$G = GL(V)$ naturally acts on $P(V)$:

$$P(V) \ni e = (e_\mu) \xrightarrow{a \in GL(V)} e' = e \cdot a^{-1}$$

$$e'_\mu = e_\nu a^{-1\,\nu}{}_\mu$$

now in

$$P(V) \times W \ni (e, w) \xrightarrow{\varphi_a} \varphi_a(e, w) = (e a^{-1}, \rho(a) w)$$

this is a left action of $GL(V)$ on $P(V) \times W$

$(e, w) \sim (e', w') \iff \exists a \in GL(n, \mathbb{R})$ s.t.

$$(e', w') = \varphi_a(e, w).$$

this is an equivalence relation in $P(V) \times W$.

Check!

$$W_\rho = P(V) \times W / \sim_\rho$$

space of objects of type ρ

One can introduce a structure of a vector space in W_ρ

$[(e, w)]$, $[(e', w')] \sim [(e, \tilde{w})]$

$$\alpha [(e, w)] + \beta [(e', w')] = [(e, \alpha w + \beta \tilde{w})]$$

Check that this does not depend on the choice of representatives

$\dim W_\rho = \dim W$

Examples

① $W = \mathbb{R}^{n(r+s)}$, $V = \mathbb{R}^n$,

$$\varrho(a) K^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} = a^{\mu_1}{}_{\alpha_1} \dots a^{\mu_r}{}_{\alpha_r} K^{\alpha_1 \dots \alpha_r}{}_{\beta_1 \dots \beta_s} a^{-1\beta_1}{}_{\nu_1} \dots a^{-1\beta_s}{}_{\nu_s}$$

$$= \varrho^r_s(a) K^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}$$

$W_\varrho = V^r_s$ tensors of type $\binom{r}{s}$

② $W = \mathbb{R}^{n(r+s)}$, $V = \mathbb{R}^n$

$$\varrho(a) = (\det a)^w \varrho^r_s(a)$$

W_ϱ - tensor densities of weight w.

e.g. $\boxed{A}$ Levi-Civita symbol defined by a) b) c):

a) $\varepsilon_{\mu_1 \dots \mu_n} = \epsilon_{[\mu_1 \dots \mu_n]}$; b) $\epsilon_{1 \dots n} = 1$

totally skew symmetric in $\underline{\underline{n}}$ indices

c) $\epsilon'_{\mu_1 \dots \mu_n} = \epsilon_{\mu_1 \dots \mu_n}$

$$a^{-1\mu_1}{}_{\nu_1} \dots a^{-1\mu_n}{}_{\nu_n} \epsilon_{\mu_1 \dots \mu_n} = (\det a)^{-1} \epsilon_{\nu_1 \dots \nu_n} =$$

$$= (\det a)^{-1} \epsilon'_{\nu_1 \dots \nu_n}$$

$$\Rightarrow \quad \epsilon'_{\nu_1 \dots \nu_n} = (\det a) \varrho_n(a) \epsilon_{\nu_1 \dots \nu_n}$$

density of covariant n-tensor of weight $\underline{+1}$.

$\boxed{B}$ $\det(g_{\mu\nu})$

$$\det(g'_{\mu\nu}) = (\det a)^{-2} \det(g_{\mu\nu})$$

scalar density of weight -2

$\boxed{C}$ $W = \mathbb{R}^{n(r+s)}$, $\varrho(a) = \operatorname{sgn}(\det a) \varrho^r_s(a)$

W_ϱ - pseudotensors

e.g.

$$\eta_{\mu_1 \dots \mu_n} = \sqrt{|\det g|}\, \epsilon_{\mu_1 \dots \mu_n}$$

Tensor fields

$$T_p(M) \rightsquigarrow T_p(M)^r_s$$

$$M \ni p \longrightarrow K_p \in T_p(M)^r_s$$

locally $(\mathcal{U}, x)$:

$$K = \underbrace{K^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}}_{\text{smooth}} \frac{\partial}{\partial x^{\mu_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\mu_r}} \otimes dx^{\nu_1} \otimes \dots \otimes dx^{\nu_s}$$

smooth $\Longrightarrow$ K is smooth

$\mathfrak{X}(M)^r_s$ – module of smooth tensor fields on M over $\mathcal{F}(M)$
vector space of smooth tensor fields on M over $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

$$\mathfrak{X}(M)^0_0 \doteq \mathcal{F}(M)$$

$$\mathcal{T}(M) = \Big(\bigoplus_{\substack{r \geq 0 \\ s \geq 0}}^{\infty} \mathfrak{X}(M)^r_s \, , \, \otimes \Big)$$

algebra of smooth tensor fields over M.